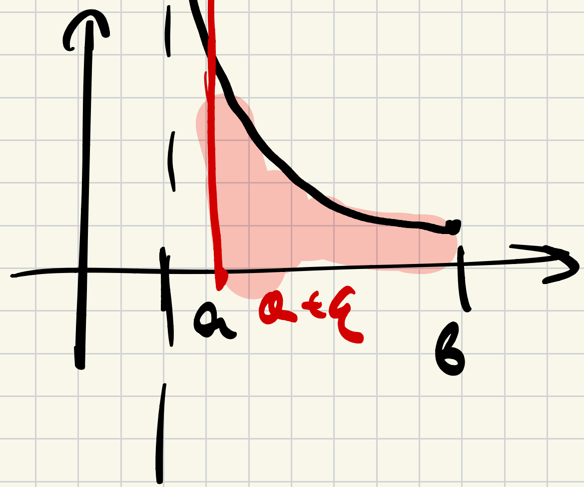
Analysis I Lecture 26

Last time: Improper integrals:

$$\int_{a^+}^{b} f(t)\,dt = \lim_{x \to a^+} \int_{x}^{b} f(t)\,dt$$

$x = a+\varepsilon$

$\varepsilon > 0$



$$\int_{a}^{b^-} f(t)\,dt = \lim_{x \to b^-} \int_{a}^{x} f(t)\,dt$$

$$\int_{a^+}^{b^-} f(t)\,dt = \int_{a^+}^{c} f(t)\,dt + \int_{c}^{b^-} f(t)\,dt$$

These definition also make sense for $a = -\infty$

$b = +\infty$

e.g.

$$\int_a^{+\infty} f(x)\,dx = \lim_{M\to\infty} \int_a^M f(x)\,dx.$$

Example

$$\int_{0^+}^{1} \log(t)\,dt =$$

$$= \lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{1} \log(t)\,dt =$$

$$= \lim_{\varepsilon \to 0^+} \left(\log(t) \cdot t - t \right) \Big|_{\varepsilon}^{1} =$$

$$\lim_{\varepsilon \to 0^+} \left(\log(t) \cdot t - t \right) \Big|_{\varepsilon}^{1} =$$

$$= \lim_{\varepsilon \to 0^+} \left(\left(\log(1) \cdot 1 - 1 \right) - \left(\log(\varepsilon) \cdot \varepsilon - \varepsilon \right) \right) =$$

$$= -1 - \lim_{\varepsilon \to 0^+} \log(\varepsilon) \cdot \varepsilon = -1.$$

Applying L'Hopital

$$\lim_{\varepsilon \to 0^+} \frac{\log(\varepsilon)}{\frac{1}{\varepsilon}} = \lim_{\varepsilon \to 0^+} \frac{(\log \varepsilon)'}{\left(\frac{1}{\varepsilon}\right)'} = \lim_{\varepsilon \to 0^+} \frac{\frac{1}{\varepsilon}}{-\frac{1}{\varepsilon^2}} = 0$$

Example

$$\int_{0^+}^{1} \frac{1}{t}\,dt =$$

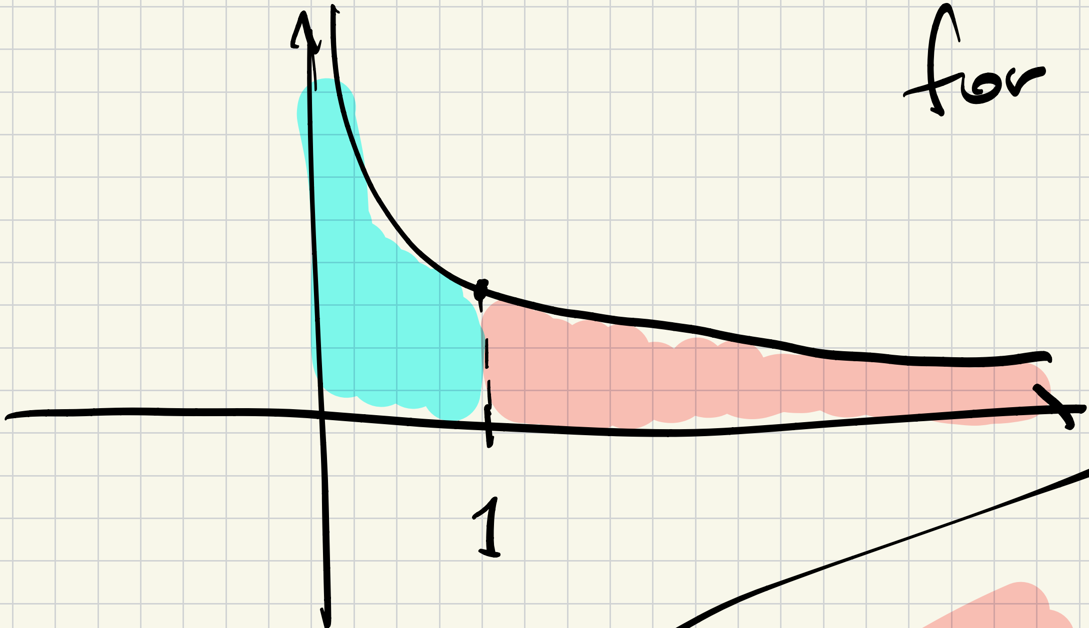
$$= \lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{1} \frac{1}{t}\,dt = \lim_{\varepsilon \to 0^+} \left(\log|t| \Big|_{\varepsilon}^{1} \right) =$$

$$= \lim_{\varepsilon \to 0^+} \left(\cancelto{0}{\log(1)} - \log(\varepsilon) \right) =$$

$$= +\infty .$$

Example

$$f_\alpha(x) = \frac{1}{x^\alpha} \qquad \alpha > 0$$



for which α these converge?

$$\int_{0^+}^{1} f_\alpha(x)\, dx \qquad\qquad \int_{1}^{\infty} f_\alpha(x)\, dx$$

1) For which $\alpha > 0$ $\int_{0^+}^{1} \frac{1}{x^\alpha} \, dx$ converges?

__Case__ $\alpha \neq 1$: the anti-derivative

of $\frac{1}{x^\alpha}$ is given by:

$$\frac{1}{1-\alpha} \cdot \frac{1}{x^{\alpha-1}}$$

So we get

$$\int_{0^+}^{1} \frac{1}{x^\alpha}\,dx = \lim_{\varepsilon \to 0^+} \frac{1}{1-\alpha} \cdot \frac{1}{x^{\alpha-1}} \Bigg|_{\varepsilon}^{1} =$$

$$= \frac{1}{1-\alpha} - \lim_{\varepsilon \to 0^+} \left(\frac{1}{1-\alpha} \cdot \frac{1}{\varepsilon^{\alpha-1}} \right)$$

$$\| \qquad \begin{cases} 0 & \text{if } \alpha - 1 < 0 \\ -\infty & \text{if } \alpha - 1 > 0 \end{cases}$$

So $$\int_{0^+}^{1} \frac{1}{x^\alpha}\,dx = \begin{cases} \frac{1}{1-\alpha} & \text{if } \alpha < 1 \\ +\infty & \text{if } \alpha > 1 \end{cases}$$

the only case left is $\alpha = 1$

We showed in a previous example that $\int_{0^+}^{1} \frac{1}{x}\,dx$ diverges.

We get

$$\int_{0^+}^{1} \frac{1}{x^\alpha}\, dx \quad \text{converges for } \alpha > 1$$

$$\text{diverges for } \alpha \leq 1$$

2) For which $\alpha > 0$ $\int_1^\infty \frac{1}{x^\alpha}\, dx$ converge?

<u>Case $\alpha \neq 1$</u>

$$\int_1^\infty \frac{1}{x^\alpha}\, dx = \lim_{M \to \infty} \int_1^M \frac{1}{x^\alpha}\, dx =$$

$$= \lim_{M \to \infty} \frac{1}{1-\alpha} \cdot \frac{1}{x^{\alpha-1}} \Bigg|_1^M =$$

$$= \lim_{M \to \infty} \frac{1}{1-\alpha} \cdot \frac{1}{x^{\alpha-1}} \Big|_1^M =$$

$$= \underbrace{\lim_{M \to \infty} \frac{1}{1-\alpha} \cdot \frac{1}{M^{\alpha-1}}}_{} - \frac{1}{1-\alpha}$$

$$\begin{cases} 0 & \alpha - 1 > 0 \\ +\infty & \alpha - 1 < 0 \end{cases}$$

So we get

$$\int_1^\infty \frac{1}{x^\alpha}\,dx = \begin{cases} \frac{1}{\alpha-1} & \alpha > 1 \\ +\infty & \alpha < 1 \end{cases}$$

Case $\alpha = 1$

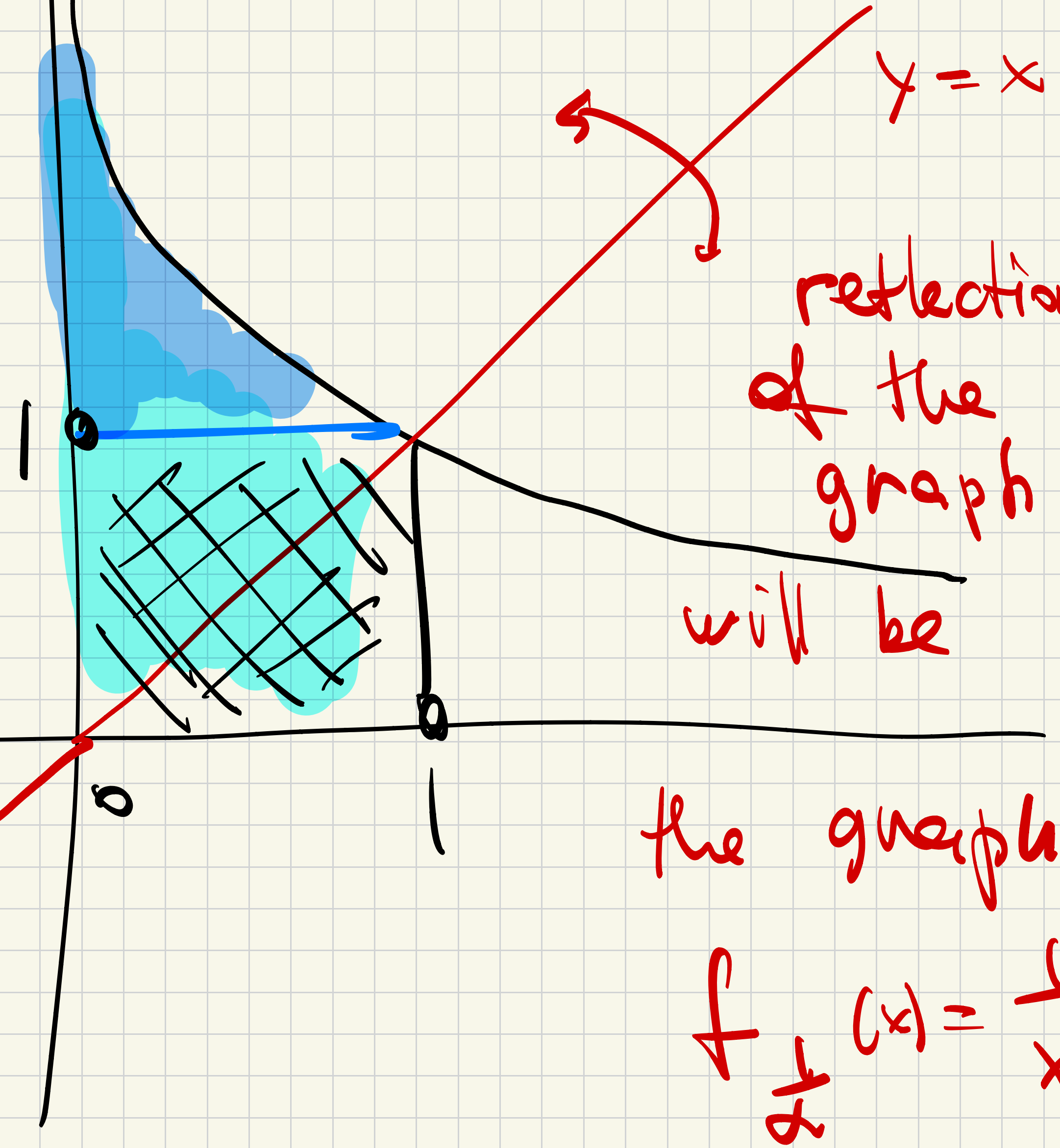
$$\int_1^\infty \frac{1}{x}\,dx = \lim_{M\to\infty} \log|x|\Big|_1^M =$$

$$= \lim_{M\to\infty} \log(M) = +\infty$$

$$\Rightarrow \int_1^\infty \frac{1}{x^\alpha}\,dx \quad \begin{array}{ll} \text{converges} & \text{if } \alpha > 1 \\ \text{diverges} & \text{if } \alpha \leq 1 \end{array}$$

$f_\alpha(x) = \frac{1}{x^\alpha}$

This shows that $\int_0^1 \frac{1}{x^\alpha}\,dx$ converges iff $\int_1^\infty \frac{1}{x^{\frac{1}{\alpha}}}\,dx$ converges



reflection of the graph will be the graph of $f_{\frac{1}{\alpha}}(x) = \frac{1}{x^{\frac{1}{\alpha}}}$

If $\alpha < 1$ then $\frac{1}{\alpha} > 1$

$$\int_{0^+}^{1} \frac{1}{x^\alpha}\,dx = \frac{1}{1-\alpha}$$

$$\int_{1}^{\infty} \frac{1}{x^{\frac{1}{\alpha}}}\,dx = \frac{1}{\frac{1}{\alpha}-1}$$ so the

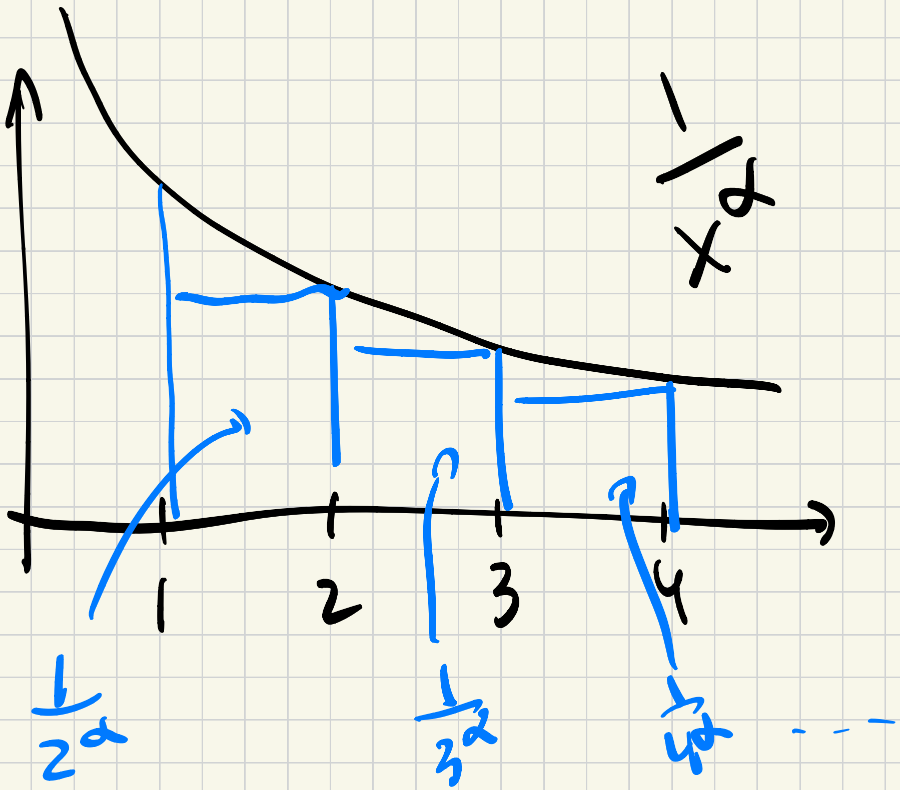
difference is $\frac{1}{\frac{1}{\alpha}-1} - \frac{1}{1-\alpha} = \frac{\alpha}{1-\alpha} - \frac{1}{1-\alpha}$

$= \frac{\alpha-1}{1-\alpha} = -1$

Corollary $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n^\alpha}$ converges

if and only if $\alpha > 1$.

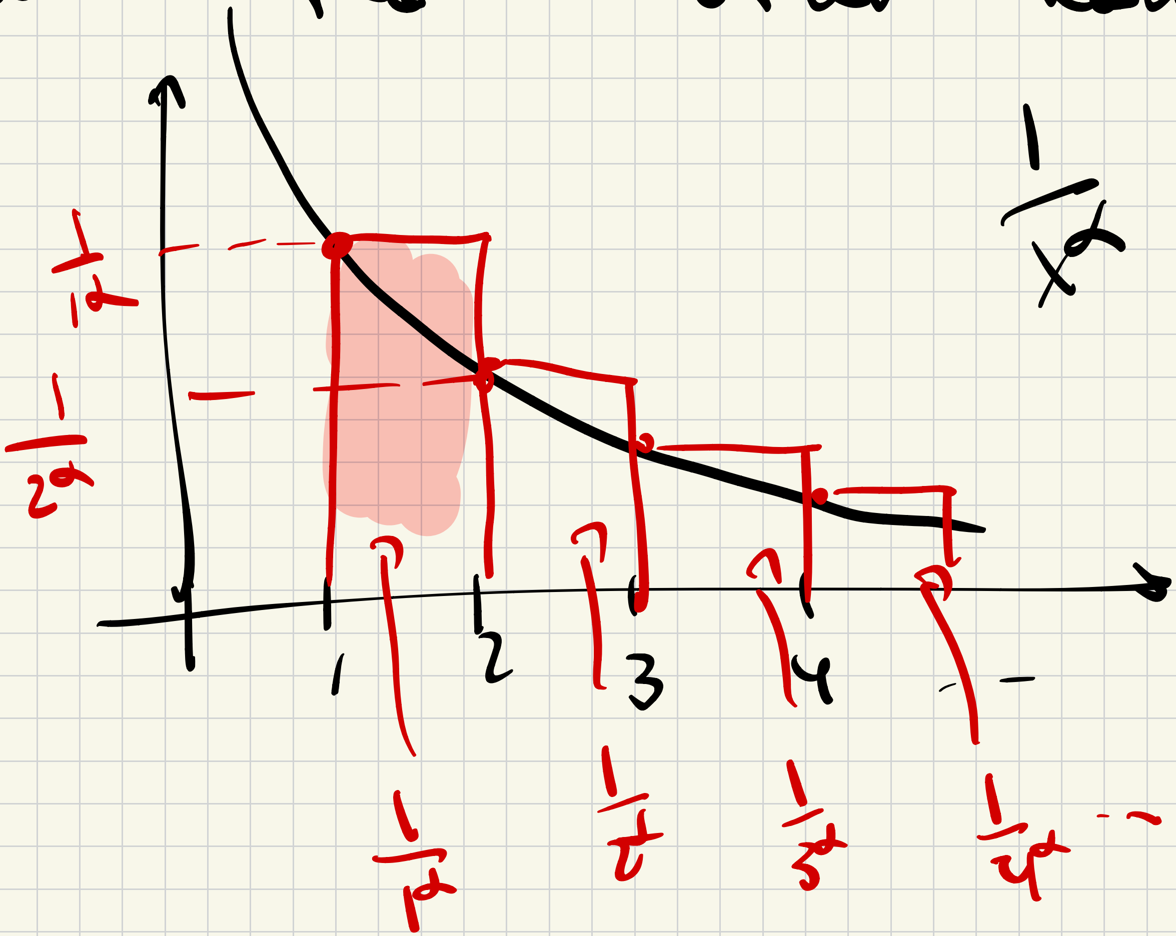
Idea:



$$\Rightarrow \sum_{n=2}^{\infty} \frac{1}{n^\alpha} \leq \int_1^\infty \frac{1}{x^\alpha}\,dx$$

On the other hand

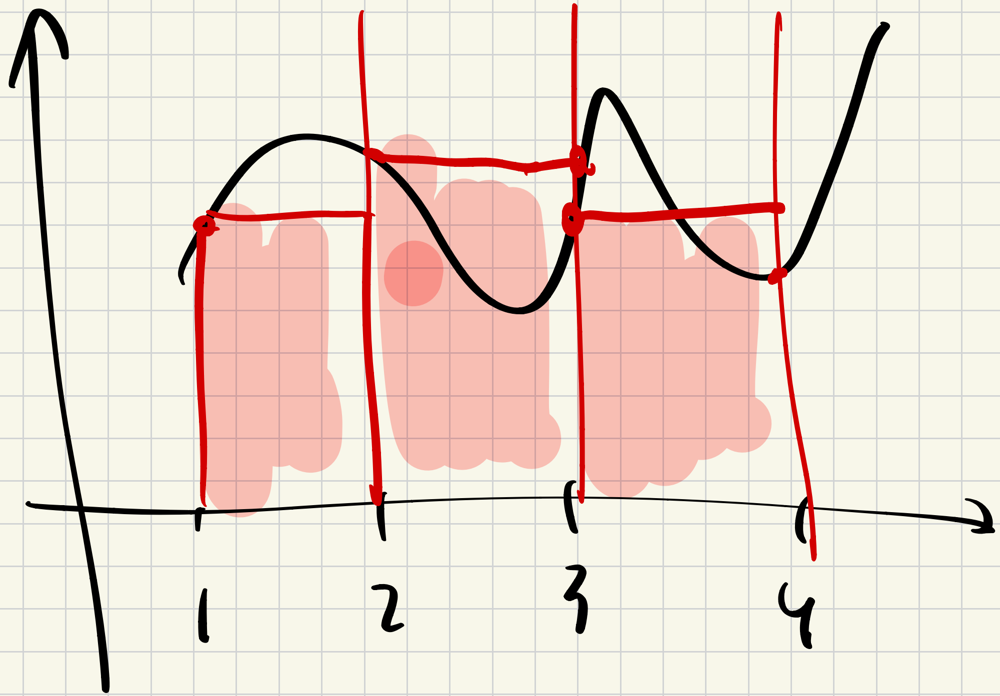
$\frac{1}{x^\alpha}$



So we get

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha} \geq \int_1^{\infty} \frac{1}{x^\alpha} dx$$

$\Rightarrow$ if integral diverges so does the series.


1
2
3
4

Some convergence criteria:

Proposition 8.57

i). If f extends continuously to $[a,b]$ with $a,b \in \mathbb{R}$ then

$$\int_{a}^{b^-} f(x)\,dx \quad , \quad \int_{a^+}^{b} f(x)\,dx \quad , \quad \int_{a^+}^{b^-} f(x)\,dx$$

converge and equal to $\int_a^b f(x)\,dx$.

- If $0 \leq f \leq g$ we get

i) if the improper integral $\int_a^b g(x)\,dx$ converges then $\int_a^b f(x)\,dx$ converges and $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$.

ii) If $\int_a^b f(x)\,dx$ diverges then $\int_a^b g(x)\,dx$ diverges.

__Remark__ in particular,

If $f(x) \geq 0$ and $\int_a^b f(x)\,dx$

converged then $\int_a^b f(x)\,dx \geq 0$

And if $f(x) \leq 0$ then

$$\int_a^b f(x)\,dx \leq 0 .$$

Definition 8.59 We say that improper integral is absolutly convergent if $\int_a^b |f|\, dx$ is convergent

Proposition 8.60 If an improper integral converges absolutely then it converges.

<u>Example</u> $\displaystyle\int_{\frac{\pi}{4}}^{+\infty} \frac{\cos(x)}{x^2}\,dx$

Notice that $\left|\dfrac{\cos(x)}{x^2}\right| \leq \dfrac{1}{x^2}$

Using that $\displaystyle\int_{\frac{\pi}{4}}^{\infty} \frac{1}{x^2}\,dx$ converges

we conclude that $\displaystyle\int_{\frac{\pi}{4}}^{+\infty} \frac{\cos(x)}{x^2}\,dx$ is

absolutely convergent $\Rightarrow$ it is convergent!

Anti-derivatives and power series

Theorem 98 Let $f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k$ be a power series with radius of convergence $R > 0$.

Then $\forall x \in (x_0 - R, x_0 + R)$ we have:

$$f'(x) = \sum_{k=1}^{\infty} k a_k (x - x_0)^{k-1}$$

$$\int_{x_0}^{x} f(t)\,dt = \sum_{k=0}^{\infty} \frac{a_k}{k+1} (x - x_0)^{k+1}$$

Remark:

Anti-derivatives are only defined up to additive constant

$$F(x) = \int_{x_0}^{x} f(t)\, dt$$ is one of them

which is characterised by $F(x_0) = 0$

In general if G is

another anti-derivative of $f(x)$

then

$$G = C + \sum_{k=0}^{\infty} \frac{a_k}{k+1} (x - x_0)^{k+1}$$

where C is some constant.

Remark One can show that if the radius of convergence of

$$f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k$$

is R then the radius of convergence of

$$\underbrace{\sum_{k=1}^{\infty} k a_k (x-x_0)^{k-1}}_{f'} \quad \text{and} \quad \sum_{k=0}^{\infty} \frac{a_k}{k+1} (x-x_0)^{k+1}$$

is also R.

So in particular we can compute for any $x \in (x_0 - R, x_0 + R)$

$$f^{(n)}(x) = \sum_{k=n}^{\infty} k(k-1)(k-2)\dots(k-n+1) \cdot a_k (x - x_0)^{k-n}$$

And in particular f is C^∞ on $(x_0 - R, x_0 + R)$.

Remark Similar formulas hold for order n expansions

E.g Let $f \in C^n(I)$ $x_0 \in I$

We have $f(x) = \sum_{k=0}^{n} a_k \cdot (x - x_0)^k + \varepsilon_n (x - x_0)^n$

where $\varepsilon_n \to 0$ as $x \to x_0$

And let G be its anti-derivative

Then $G(x) = C + \sum_{k=0}^{n} \frac{a_k}{k+1} (x - x_0)^{k+1} + \varepsilon_{n+1} (x - x_0)^{n+1}$

$$= C + \sum_{k=1}^{n+1} \frac{a_{k-1}}{k} (x - x_0)^k + \varepsilon_{n+1} (x - x_0)^{n+1}$$

Example: Compute Taylor expansion of $\arctan(x)$.

Idea $\arctan(x)' = \frac{1}{1+x^2} = f(x^2)$

where $f(x) = \frac{1}{1+x}$

$$T_{\frac{1}{1+x},0}(x) = \sum_{k=0}^{\infty} (-1)^k x^k \implies$$

$$T_{\frac{1}{1+x^2},0}(x) = \sum_{k=0}^{\infty} (-1)^k x^{2k}.$$

So we get that

$$\arctan'(x) = \sum_{k=0}^{\infty} (-1)^k x^{2k}$$

$$\Rightarrow \arctan(x) = C + \sum_{k=0}^{+\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$$

How to find C? $\arctan(0) = 0$

$\Rightarrow C = 0$ $\blacksquare$

Example Compute Taylor series

of $f(x) = \int_0^x e^{-t^2}\,dt$

First $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow$

$\Rightarrow e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-t^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{n!}$

Therefore the anti-derivative $f(x)$ is given by:

$$f(x) = C + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \cdot \frac{x^{2k+1}}{2k+1}$$

Since $f(0) = 0$ we get $C = 0$ ∎

The end!